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Symmetric functions and B_N-invariant spherical harmonics

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Abstract
The wavefunctions of a quantum isotropic harmonic oscillator modified by reflecting barriers at the coordinate planes in N-dimensional space can be expressed in terms of certain generalized spherical harmonics. These are associated with a product-type weight function on the sphere. Their analysis is carried out by means of differential-difference operators. The symmetries of this system involve the Weyl group of type B, generated by permutations and changes of sign of the coordinates. A new basis for symmetric functions as well as an explicit transition matrix to the monomial basis is constructed. This basis leads to a basis for invariant spherical harmonics. The determinant of the Gram matrix for the basis in the natural inner product over the sphere is evaluated. When the underlying parameter is specialized to zero, the basis consists of ordinary spherical harmonics with cube group symmetry, as used for wavefunctions of electrons in crystals. The harmonic oscillator can also be considered as a degenerate interaction-free spin Calogero model.

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1. Introduction

There are interesting families of potentials invariant under permutations and change of signs of coordinates. The most basic one is a central potential perturbed by a crystal field with cubic symmetry. Another important example is the spin Calogero–Moser system of B type. In this paper, we study a potential which can be considered as an N-dimensional isotropic harmonic oscillator modified by barriers at the coordinate hyperplanes, or as a degenerate Calogero–Moser model with no interaction. The main object is to study invariant harmonic polynomials, which when multiplied by radial Laguerre polynomials provide a complete decomposition of the invariant wavefunctions. This is made possible by use of the author's differential-difference operators and the construction of a new basis for symmetric functions.

We consider generalized spherical harmonic polynomials invariant under the action of the hyperoctahedral group B_N acting on $\mathbb{R}^N$. Let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, the set of compositions with N parts is $\mathbb{N}_0^N$; for $\alpha \in \mathbb{N}_0^N$ let $|\alpha| = \sum_{i=1}^N \alpha_i$, $\alpha! = \prod_{i=1}^N \alpha_i!$ and $(t)_\alpha = \prod_{i=1}^N (t)_{\alpha_i}$, (the Pochhammer symbol is $(t)_n = \prod_{i=1}^n (t + i - 1)$). Denote the cardinality of a finite set E by $\#E$. The set $\mathcal{P}$ of partitions consists of finite sequences $\lambda = (\lambda_1, \lambda_2, \ldots) \in \bigcup_{N=1}^\infty \mathbb{N}_0^N$ satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$ and two partitions with the same nonzero components are identified. Then $\mathcal{P}_n = \{\lambda \in \mathcal{P} : |\lambda| = n\}$ for $n = 1, 2, 3, \ldots$. For a partition the length $l(\lambda) = \#\{i : \lambda_i \geqslant 1\}$ is the number of nonzero components. For a given N we will need the partitions with $l(\lambda) \leqslant N$, denoted by $\mathcal{P}^{(N)} = \mathcal{P} \cap \mathbb{N}_0^N$, and $\mathcal{P}_n^{(N)} = \mathcal{P}_n \cap \mathbb{N}_0^N$. One partial ordering for partitions is given by containment of Ferrers diagrams: thus $\mu \subset \lambda$ means $\mu_i \leqslant \lambda_i$ for each i. The notation $\varepsilon_i = (0, \ldots, \overset{i}{1}, \ldots)$ for the ith standard basis vector in $\mathbb{N}_0^N$ (for $1 \leqslant i \leqslant N$) is convenient for describing contiguous partitions; for example $\lambda + \varepsilon_1 = (\lambda_1 + 1, \lambda_2, \ldots)$.

For $x \in \mathbb{R}^N$ and $\alpha \in \mathbb{N}_0^N$ let $x^\alpha = \prod_{i=1}^N x_i^{\alpha_i}$, a monomial of degree $|\alpha|$. Let $\mathbb{P}_n^{(N)} = \operatorname{span}\{x^\alpha : \alpha \in \mathbb{N}_0^N, |\alpha| = n\}$, the space of homogeneous polynomials of degree $n \geqslant 0$ in N variables. The Laplacian is $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$, and the Euclidean norm is $\|x\| = \left(\sum_{i=1}^N x_i^2\right)^{1/2}$. Fix the parameter $\kappa \geqslant 0$.

We will be concerned with the operator $\Delta + 2\kappa \sum_{i=1}^N \frac{1}{x_i} \frac{\partial}{\partial x_i}$. It is associated with the Calogero–Sutherland model with the potential function $V(x) = \omega^2 \|x\|^2 + \kappa(\kappa - 1) \sum_{i=1}^N \frac{1}{x_i^2}$ (with $\omega > 0$). This is an N-dimensional isotropic harmonic oscillator modified by barriers at the coordinate hyperplanes; or it could be considered as a degenerate form of the type-B spin model for N particles with no interaction. The spin model with interactions and reflecting barriers was studied by Yamamoto and Tsuchiya [11]. With the base state

$$\psi(x) = \exp\left(-\frac{\omega}{2}\|x\|^2\right) \prod_{i=1}^N |x_i|^\kappa$$

we obtain the conjugate of the Hamiltonian, for any smooth function f on $\mathbb{R}^N$

$$\psi(x)^{-1} \left(-\Delta + \omega^2 \|x\|^2 + \kappa(\kappa - 1) \sum_{i=1}^N \frac{1}{x_i^2}\right) (\psi(x) f(x))$$
$$= \left(-\Delta - 2\kappa \sum_{i=1}^N \frac{1}{x_i} \frac{\partial}{\partial x_i} + N\omega(2\kappa + 1) + 2\omega \sum_{i=1}^N x_i \frac{\partial}{\partial x_i}\right) f(x).$$

The $\mathbb{Z}_2^N$ version of the differential-difference operators introduced by the author ([2], or see [5, ch 4]) are the fundamental tools for analysing this eigenfunction problem. The action of the group B_N on $\mathbb{R}^N$ induces an action on functions, denoted by $wf(x) = f(xw)$ for $w \in B_N$. For $1 \leqslant i \leqslant N$ let σ_i denote the reflection

$$(x_1, \ldots, x_N)\sigma_i = (x_1, \ldots, -x_i, \ldots, x_N)$$

and define the first-order operator $\mathcal{D}_i$ by

$$\mathcal{D}_i f(x) = \frac{\partial}{\partial x_i} f(x) + \kappa \frac{f(x) - f(x\sigma_i)}{x_i}$$

for sufficiently smooth functions f on $\mathbb{R}^N$. The Laplacian operator is

$$\Delta_\kappa = \sum_{i=1}^N \mathcal{D}_i^2$$

then

$$\Delta_\kappa f(x) = \Delta f(x) + \kappa \sum_{i=1}^{N} \left(\frac{2}{x_i} \frac{\partial}{\partial x_i} f(x) - \frac{f(x) - f(x\sigma_i)}{x_i^2} \right).$$

If f is even in each x_i (that is, $\mathbb{Z}_2^N$-invariant) then $\Delta_\kappa f(x) = \Delta f(x) + 2\kappa \sum_{i=1}^{N} \frac{1}{x_i} \frac{\partial}{\partial x_i} f(x)$. If we modify the above Hamiltonian to

$$\mathcal{H} = -\Delta + \omega^2 \|x\|^2 + \kappa \sum_{i=1}^{N} \frac{\kappa - \sigma_i}{x_i^2} \tag{1.1}$$

which is identical to the previous one when acting on $\mathbb{Z}_2^N$-invariant functions, we obtain

$$\psi^{-1}\mathcal{H}\psi f(x) = \left(-\Delta_\kappa + 2\omega \sum_{i=1}^{N} x_i \frac{\partial}{\partial x_i} + \omega N(2\kappa + 1) \right) f(x).$$

The eigenfunctions of this operator can be expressed as products of radial Laguerre polynomials and homogeneous harmonic polynomials.

Definition 1. *For $n \geqslant 0$ the space of harmonic (sometimes called h-harmonic) homogeneous polynomials of degree n is $\mathbb{H}_n = \left\{ f \in \mathbb{P}_n^{(N)} : \Delta_\kappa f = 0 \right\}$. Let $\mathbb{H}_{2n}^0 = \{ f \in \mathbb{H}_{2n} : \sigma_i f = f, 1 \leqslant i \leqslant N \}$; this is the space of $\mathbb{Z}_2^N$-invariant harmonic polynomials, even in each x_i, $1 \leqslant i \leqslant N$. The subspace of $\mathbb{H}_{2n}^0$ consisting of polynomials invariant under permutation of coordinates is denoted by $\mathbb{H}_{2n}^B$.*

It is not hard to show that if $f \in \mathbb{H}_n$ then $L_s^{(n+N/2+N\kappa-1)}(\omega\|x\|^2) f(x)$ is an eigenfunction of $\psi^{-1}\mathcal{H}\psi$ with eigenvalue $\omega(2n + 4s + N(2\kappa + 1))$ (this calculation uses equation (3.1)). So combining bases for each $\mathbb{H}_n$ with Laguerre polynomials provides a basis for the polynomial eigenfunctions (positive energy) of $\psi^{-1}\mathcal{H}\psi$. The use of functions of this type in the general Calogero–Moser models of types A and B is discussed by van Diejen [1]. In Cartesian coordinates the eigenfunctions of $\psi^{-1}\mathcal{H}\psi$ can be expressed as products of generalized Hermite polynomials (see [5, p 25]), namely $\prod_{i=1}^{N} H_{\alpha_i}^{\kappa}(\omega^{1/2} x_i)$, $\alpha \in \mathbb{N}_0^N$. More details on the B_N-invariant basis produced by these polynomials are given in the last section.

The classical motion problem corresponding to $\mathcal{H}$ is easily solved: in the one-dimensional case the particle with mass 1 at $s \in \mathbb{R}$ satisfies $\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 s(t) = -\frac{\partial}{\partial s} V(s) = -2\omega^2 s + 2\kappa(\kappa - 1)s^{-3}$ with the solution (for $s > 0$, $\kappa > 1$)

$$s(t) = (q + a \sin(2^{3/2}\omega(t - t_0)))^{1/2} \qquad q = (a^2 + \kappa(\kappa - 1)/\omega^2)^{1/2}$$

where t_0 is an arbitrary phase shift, $a \geqslant 0$ is arbitrary and the energy is $\frac{1}{2}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 + V(s) = 2q\omega^2$.

Primarily we will study the invariant functions associated with Δ_κ; the invariance is with respect to the Weyl group of type B, thus the functions are invariant under sign changes $\{\sigma_i\}$ and permutations of coordinates. Such functions are expressed as symmetric functions of the variable $x^2 = \left(x_1^2, \ldots, x_N^2\right)$. Note that a special case of this study is the problem of spherical harmonics on $\mathbb{R}^3$ which satisfy B_3-invariance, appearing in the wavefunctions of electrons in crystals. We will construct an explicit basis for these polynomials, but unfortunately it is not orthogonal. There seems to be a good reason why an orthogonal basis has not yet been found: let $R_{ij} = x_i \mathcal{D}_j - x_j \mathcal{D}_i$ (then $R_{ij}\Delta_\kappa = \Delta_\kappa R_{ij}$, and $\sqrt{-1}R_{ij}$ is an angular momentum operator), and let $\mathcal{S}_n = \sum_{1 \leqslant i < j \leqslant N} R_{ij}^{2n}$, then $\mathcal{S}_1$ is the Casimir (or Laplace–Beltrami) operator, for each n the operator $\mathcal{S}_n$ is self-adjoint in the natural inner product on the sphere (defined later) but $\mathcal{S}_2, \mathcal{S}_3$ do not commute, already for $N = 3$. So the usual machinery for constructing good orthogonal bases (such as Jack polynomials) does not work here. (The exploratory phase of this paper involved computing with all of the plausible B_N-commuting self-adjoint operators of degrees 2, 4, 6; none of them had rational eigenvalues (in $\mathbb{Q}(\kappa)$) which would be necessary for explicit formulae.) The conjugate

$$\psi R_{ij}\psi^{-1} = x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i} - \kappa\left(\frac{x_i}{x_j}\sigma_j - \frac{x_j}{x_i}\sigma_i\right)$$

commutes with the Hamiltonian $\mathcal{H}$ (see Taniguchi [9] for a more general treatment of commuting operators in the context of r^{-2}-type potentials). By straightforward calculation

$$\mathcal{S}_1 = \sum_{1\leqslant i<j\leqslant N} R_{ij}^2 = \|x\|^2\Delta_\kappa - \left(\sum_{i=1}^N x_i\mathcal{D}_i\right)^2 - \left(N-2+2\kappa\sum_{i=1}^N \sigma_i\right)\left(\sum_{i=1}^N x_i\mathcal{D}_i\right).$$

Thus $\mathcal{S}_1$ has the eigenvalue $-2n(N-2+2n+2N\kappa)$ on $\mathbb{H}_{2n}^0$.

2. A basis for symmetric functions

In this section the number N of variables is not specified, with the understanding that it is not less than the length of any partition that appears. We use the notation of Macdonald [8] for symmetric polynomials in the variable $x \in \mathbb{R}^N$. Let S_N denote the symmetric group on N objects, considered as the group of $N \times N$ permutation matrices, acting on the left on $\mathbb{N}_0^N$. For any $\alpha \in \mathbb{N}_0^N$ let α^+ denote the unique partition $w\alpha \in \mathcal{P}^{(N)}$, for some $w \in S_N$ (the sorting of the components of α in nonincreasing order; w need not be unique).

Definition 2. *For $\lambda \in \mathcal{P}^{(N)}$ the monomial symmetric function is*

$$m_\lambda = \sum\left\{x^\alpha : \alpha \in \mathbb{N}_0^N, \alpha^+ = \lambda\right\}$$

summing over all distinct permutations of λ. The elementary symmetric function of degree 1 is

$$e_1 = \sum_{i=1}^N x_i.$$

It turns out that the basis elements for invariant harmonics are labelled by partitions with the property $\lambda_1 = \lambda_2$; that is, $\dim \mathbb{H}_{2n}^B = \#\left\{\lambda \in \mathcal{P}_n^{(N)} : \lambda_1 = \lambda_2\right\}$. Further the formula for the projection onto harmonics uses powers of $\|x\|^2 = e_1\left(x_1^2, \ldots, x_N^2\right)$. This leads to the following definition of a basis for symmetric functions well suited for the present study.

Definition 3. *For $\lambda \in \mathcal{P}^{(N)}$ the modified monomial symmetric function is*

$$\widetilde{m}_\lambda = e_1^{\lambda_1-\lambda_2} m_{(\lambda_2,\lambda_2,\lambda_3,\ldots)}.$$

We will show that $\{\widetilde{m}_\lambda : \lambda \in \mathcal{P}^{(N)}\}$ is a basis for the symmetric polynomials on $\mathbb{R}^N$ and the transition matrix to the m-basis has entries in $\mathbb{Z}$, is unimodular and triangular in a certain ordering. One direction is an easy consequence of the multinomial theorem. We use the notation $\binom{j}{\alpha}$ for the multinomial coefficient, where $\alpha \in \mathbb{Z}^N$, $\sum_{i=1}^N \alpha_i = j$, and $\binom{j}{\alpha} = \frac{j!}{\alpha!}$ if each $\alpha_i \geqslant 0$, else $\binom{j}{\alpha} = 0$.

Definition 4. *For $\lambda, \nu \in \mathcal{P}^{(N)}$ the coefficient $\left\langle {\lambda \atop \nu} \right\rangle$ is defined by the expansion*

$$e_1^j m_\nu = \sum_{|\lambda|=|\nu|+j} \left\langle {\lambda \atop \nu} \right\rangle m_\lambda$$

for $j = 0, 1, 2, \ldots$.

We establish some basic properties as well as an explicit formula for this coefficient, and also explain the relation with the generalized binomial coefficient.

Proposition 1. *For $\lambda, \nu \in \mathcal{P}^{(N)}$ the following hold:*

(1) let $j = |\lambda| - |\nu| > 0$, then

$$\left\langle {\lambda \atop \nu} \right\rangle = \sum \left\{ \binom{j}{\lambda - \sigma} : \sigma^+ = \nu \right\}.$$

Note $\binom{j}{\lambda-\sigma} = 0$ if any component $\lambda_i - \sigma_i < 0$, the sum is over all distinct permutations of ν, (including any possible zero components to make ν an element of $\mathbb{N}_0^N$);

(2) $\left\langle {\lambda \atop \nu} \right\rangle \neq 0$ if and only if $\nu \subset \lambda$;

(3) $\left\langle {\lambda \atop \lambda} \right\rangle = 1$.

Proof. Expanding $e_1^j m_\nu$ we obtain

$$e_1^j m_\nu = \sum \left\{ \binom{j}{\alpha} x^\alpha x^\sigma : \alpha \in \mathbb{N}_0^N, |\alpha| = j, \sigma^+ = \nu \right\}.$$

Since this is a symmetric polynomial, the coefficient of m_λ equals the coefficient of x^λ, namely the sum of $\binom{j}{\alpha}$ with $\alpha + \sigma = \lambda$ (component-wise addition). For any σ with $\sigma^+ = \nu$ if $\sigma_i > \lambda_i$ for some i then the corresponding term is zero, by definition of the multinomial coefficient. For part (2), $\nu \subset \lambda$ (and $|\lambda| = |\nu| + j$) implies $\binom{j}{\lambda-\nu} > 0$; conversely if $\binom{j}{\lambda-\sigma} \neq 0$ for some σ with $\sigma^+ = \nu$ then $\nu \subset \lambda$ (if $\sigma_i \leqslant \lambda_i$ for each i then $\sigma^+ \subset \lambda$; indeed, let w be a permutation so that $\nu_i = \sigma_{w(i)}$ for each i, so that $\nu_j \leqslant \nu_i = \sigma_{w(i)} \leqslant \lambda_{w(i)}$ for $1 \leqslant i \leqslant j$ then $\nu_j \leqslant \lambda_j$ because it is less than or equal to at least j components of the partition λ). Part (3) is trivial. □

The coefficients $\left\langle {\lambda \atop \nu} \right\rangle$ are a modification of the generalized binomial coefficients which have appeared in several contexts such as Jack polynomials, see Lassalle [7]; part (2) of proposition 1 is a special case of his results. There are useful formulae involving contiguous partitions (if $\nu \supset \lambda$ and $|\nu| = |\lambda| + 1$ then $\nu = \lambda + \varepsilon_i$ for $i = 1$ or $\lambda_{i-1} > \lambda_i$).

Proposition 2. *Suppose $\lambda, \sigma \in \mathcal{P}$ and $|\sigma| \geqslant |\lambda| + 1$, also $j = 1$ or $\lambda_{j-1} > \lambda_j$ then*

$$\left\langle {\lambda + \varepsilon_j \atop \lambda} \right\rangle = 1 + \#\{i : \lambda_i = \lambda_j + 1\} \tag{2.1}$$

$$\left\langle {\sigma \atop \lambda} \right\rangle = \sum \left\{ \left\langle {\sigma \atop \lambda + \varepsilon_i} \right\rangle \left\langle {\lambda + \varepsilon_i \atop \lambda} \right\rangle : i = 1 \text{ or } \lambda_{i-1} > \lambda_i \right\}. \tag{2.2}$$

Proof. Suppose $\lambda_{s-1} > \lambda_s = \lambda_{j-1} = \lambda_j + 1$ (or $s = 1$, or $j = 1$), then there are exactly $j - s + 1$ distinct permutations $w\lambda$ for which $\lambda + \varepsilon_j - w\lambda$ has no negative components (namely the transpositions $w = (i, j)$ for some i with $s \leqslant i \leqslant j - 1$, or $w = 1$). Each such term contributes 1 to the sum defining $\left\langle {\lambda+\varepsilon_j \atop \lambda} \right\rangle$. For the second part, let $n = |\sigma| - |\lambda|$ and consider the coefficient of m_σ in

$$e_1^n m_\lambda = \sum_{|\nu|=|\lambda|+n} \left\langle {\nu \atop \lambda} \right\rangle m_\nu = e_1^{n-1} \sum_i \left\langle {\lambda+\varepsilon_i \atop \lambda} \right\rangle m_{\lambda+\varepsilon_i}$$
$$= \sum_{|\nu|=|\lambda|+n} \sum_i \left\langle {\nu \atop \lambda+\varepsilon_i} \right\rangle \left\langle {\lambda+\varepsilon_i \atop \lambda} \right\rangle m_\nu$$

summing over $i = 1$ and $i \geqslant 2$ with $\lambda_{i-1} > \lambda_i$. □

We are ready to consider the transition matrices between the m_λ and $\widetilde{m}_\lambda$ bases (that the second set is a basis will be immediately proved).

Definition 5. *The transition matrices* $A(\mu, \lambda), B(\mu, \lambda)$ (*for* $\lambda, \mu \in \mathcal{P}, |\lambda| = |\mu|$) *are defined by*

$$\widetilde{m}_\lambda = \sum_\mu A(\mu, \lambda) m_\mu \qquad m_\lambda = \sum_\mu B(\mu, \lambda) \widetilde{m}_\mu.$$

By definition 4 it follows that

$$A(\mu, \lambda) = \left\langle {\mu \atop (\lambda_2, \lambda_2, \lambda_3, \ldots)} \right\rangle.$$

The condition $A(\mu, \lambda) \neq 0$ leads to the following ordering of partitions.

Definition 6. *For* $\lambda, \mu \in \mathcal{P}$ *the relation* $\mu \preceq \lambda$ *means that* $|\lambda| = |\mu|$ *and* $\lambda_i \leqslant \mu_i$ *for all* $i \geqslant 2$.

It is clear that $\preceq$ is a partial ordering, $\mu \preceq \lambda$ implies $\mu_1 \leqslant \lambda_1, l(\lambda) \leqslant l(\mu)$ and that the maximum element of $\mathcal{P}_n^{(N)}$ is (n). In fact $\mu \preceq \lambda$ implies that λ dominates μ, but $\preceq$ is not identical to the dominance ordering. Also $\preceq$ is distinct from the reverse lexicographic ordering (see [8, p 6]), for example the partitions (3, 1, 1, 1) and (2, 2, 2) are comparable in both forms of reverse-lex but not in $\preceq$. Further the partitions $\{\lambda : \lambda_1 = \lambda_2\}$ are minimal elements; indeed suppose $\mu \preceq \lambda$ and $\lambda_1 = \lambda_2$, then $\mu_1 \leqslant \lambda_1 = \lambda_2 \leqslant \mu_2$ which implies $\mu_1 = \mu_2 = \lambda_1$; thus $\sum_{i\geqslant 2}(\mu_i - \lambda_i) = 0$ and $\lambda = \mu$ (by definition, $\mu_i - \lambda_i \geqslant 0$ for each $i \geqslant 2$). There are exceptional minimal elements of the form $(m+1, m, \ldots, m)$, with $n = Nm + 1$, but these are minimal only for $\mathcal{P}^{(N)}$ since $(m, m, \ldots, m, 1) \preceq (m+1, m, \ldots, m, 0)$. We see that the matrix $A(\mu, \lambda)$ is triangular ($A(\mu, \lambda) \neq 0$ implies $\mu \preceq \lambda$), unipotent ($A(\lambda, \lambda) = 1$) and its entries are non-negative integers.

Proposition 3. *For* $n \geqslant 0$ *the set* $\left\{\widetilde{m}_\lambda : \lambda \in \mathcal{P}_n^{(N)}\right\}$ *is a basis for the symmetric polynomials of degree n in N variables. The transition matrix* $B(\mu, \lambda)$ *is triangular for the ordering* $\preceq$ *and is unipotent with integer entries.*

Regarding the dependence on N: begin by assuming that $n \leqslant N$ so that $l(\lambda) \leqslant N$ for each $\lambda \in \mathcal{P}_n$ and the defining equations for $A(\mu, \lambda), B(\mu, \lambda)$ are unambiguous, then to restrict to a smaller number of variables, say $M < N$, substitute $x_{M+1} = x_{M+2} = \cdots = x_N = 0$ with the effect of removing all terms with $l(\lambda) > M$ or $l(\mu) > M$. Thus the transition matrices for the case of M variables with $n > M$ are principal submatrices of the general transition matrices (deleting rows and columns labelled by λ with $l(\lambda) > M$). If the index set $\mathcal{P}_n^{(N)}$ is ordered first in decreasing order of $l(\lambda)$ and then with respect to $\preceq$ (possible, since $\mu \preceq \lambda$ implies $l(\mu) \geqslant l(\lambda)$) then the triangularity makes it obvious that the submatrix of B is the inverse of the submatrix of A; see the example of B for $\mathcal{P}_6^{(4)}$ at the end of this section.

It remains to establish an explicit formula for $B(\mu, \lambda)$. The formula itself was postulated based on computer algebra experimentation, and explicit known formulae for $N = 3$. The proof will be by induction on n and requires showing that the claimed formula satisfies the following contiguity relations.

Proposition 4. *For $\lambda, \mu \in \mathcal{P}_n^{(N)}$, and $\sigma \in \mathcal{P}_{n+1}^{(N)}$ with $\sigma_1 = \sigma_2$,*

$$\sum_{\nu \supset \lambda, |\nu|=|\lambda|+1} \left\langle {\nu \atop \lambda} \right\rangle B(\mu + \varepsilon_1, \nu) = B(\mu, \lambda) \tag{2.3}$$

$$\sum_{\nu \supset \lambda, |\nu|=|\lambda|+1} \left\langle {\nu \atop \lambda} \right\rangle B(\sigma, \nu) = 0. \tag{2.4}$$

Proof. Multiply both sides of the equation $m_\lambda = \sum_{\mu \preceq \lambda} B(\mu, \lambda) \widetilde{m}_\mu$ by e_1 to obtain

$$\sum_{\nu \supset \lambda, |\nu|=|\lambda|+1} \left\langle {\nu \atop \lambda} \right\rangle m_\nu = \sum_{\mu \preceq \lambda} B(\mu, \lambda) \widetilde{m}_{\mu+\varepsilon_1} = \sum_{\nu \supset \lambda, |\nu|=|\lambda|+1} \left\langle {\nu \atop \lambda} \right\rangle \sum_{\tau \preceq \nu} B(\tau, \nu) \widetilde{m}_\tau.$$

Since the set $\left\{\widetilde{m}_\tau : \tau \in \mathcal{P}_{n+1}^{(N)}\right\}$ is a basis the desired equations are consequences of matching the coefficients in the two right-hand side expansions. □

Lemma 1. *Suppose $B'(\mu, \lambda)$ is a matrix satisfying equations (2.3) and (2.4) with B replaced by B', has the same triangularity property as B and $B'(\sigma, \sigma) = 1$ for each $\sigma \in \mathcal{P}$ with $\sigma_1 = \sigma_2$, then $B'(\mu, \lambda) = B(\mu, \lambda)$ for all $\mu, \lambda \in \mathcal{P}$ (with $|\lambda| = |\mu|$).*

Proof. This is a double induction on $|\lambda|$ and $\lambda_1 - \lambda_2$. For any $\sigma \in \mathcal{P}$ with $\sigma_1 = \sigma_2$ the hypothesis shows that $B'(\mu, \sigma) = 0$ for $\mu \neq \sigma$ (since σ is minimal) and $B'(\sigma, \sigma) = 1$, thus $B'(\mu, \sigma) = B(\mu, \sigma)$ for all μ (with $|\mu| = |\sigma|$). Suppose that $B'(\mu, \lambda) = B(\mu, \lambda)$ for all $\mu, \lambda \in \mathcal{P}$ with $|\lambda| = |\mu| = n$ or with $|\lambda| = |\mu| = n + 1$ and $\lambda_1 - \lambda_2 \leqslant j$ for some $j \geqslant 0$. Fix $\nu \in \mathcal{P}$ with $|\nu| = n + 1$ and $\nu_1 - \nu_2 = j + 1$ and let $\lambda = \nu - \varepsilon_1$. Then $\tau \supset \lambda$ and $|\tau| = n + 1$ imply $\tau = \nu$ or $\tau = \lambda + \varepsilon_i$ with $i \geqslant 2$ and $\lambda_{i-1} > \lambda_i = \nu_i$. Note that $\left\langle {\lambda+\varepsilon_i \atop \lambda} \right\rangle = 1$ by equation (2.1). For any $\mu \in \mathcal{P}_{n+1}$, by hypothesis

$$B'(\mu, \nu) = B'(\mu - \varepsilon_1, \lambda) - \sum_{i \geqslant 2, \lambda_{i-1} > \lambda_i} \left\langle {\lambda + \varepsilon_i \atop \lambda} \right\rangle B'(\mu, \lambda + \varepsilon_i)$$

replacing $B'(\mu - \varepsilon_1, \lambda)$ by 0 if $\mu_1 = \mu_2$. By the inductive hypothesis each term $B'(\sigma, \tau)$ appearing on the right-hand side satisfies $B'(\sigma, \tau) = B(\sigma, \tau)$. By the proposition, $B'(\mu, \nu) = B(\mu, \nu)$. This completes the induction. □

In the following formula the coefficient $\left\langle {\sigma \atop \tau} \right\rangle$ is used with partitions whose first parts are deleted, also $(\mu_2 - 1, \mu_3, \ldots)$ is not a partition if $\mu_2 = \mu_3$ and so the α^+ notation is used; the binomial coefficient $\binom{-1}{m} = 0$.

Theorem 1. *For $\mu, \lambda \in \mathcal{P}_n$*

$$B(\mu, \lambda) = (-1)^{\lambda_1 - \mu_1} \times \left\{ \binom{\lambda_1 - \mu_2}{\mu_1 - \mu_2} \left\langle {(\mu_2, \mu_3, \ldots) \atop (\lambda_2, \lambda_3, \ldots)} \right\rangle + \binom{\lambda_1 - \mu_2 - 1}{\mu_1 - \mu_2} \left\langle {(\mu_2 - 1, \mu_3, \ldots)^+ \atop (\lambda_2, \lambda_3, \ldots)} \right\rangle \right\}.$$

Proof. The proof is broken down into cases depending on the values of $\lambda_1 - \mu_1$ and $\mu_1 - \mu_2$. We let $B'(\mu, \lambda)$ denote the right-hand side of the stated formula and proceed as in the lemma to show that B' satisfies equations (2.3) and (2.4). It is clear that $B'(\mu,\lambda)\neq 0$ implies $\lambda_i\leqslant\mu_i$ for all $i\geqslant 2$, that is, $\mu\preceq\lambda$. Also $B'(\mu,\mu)=1$ for each $\mu\in\mathcal{P}_n$. Fix $\mu,\lambda\in\mathcal{P}_n$ and let $\mu'=(\mu_2,\mu_3,\ldots)$, $\mu''=(\mu_2-1,\mu_3,\ldots)^+$, $\sigma'=(\sigma_2,\sigma_3,\ldots)$, $\sigma''=(\sigma_2-1,\sigma_3,\ldots)^+$, $\lambda'=(\lambda_2,\lambda_3,\ldots)$ (these are partitions, but with components labelled by $i\geqslant 2$; for example $\lambda'+\varepsilon_2=(\lambda_2+1,\lambda_3,\ldots)$). The partitions ν satisfying $\nu\supset\lambda$ and $|\nu|=|\lambda|+1$ are of the form $\lambda+\varepsilon_i$ with $i=1$ or $\lambda_{i-1}>\lambda_i$. We rewrite equation (2.3) (not yet proved!) as

$$\sum_{i\geqslant 2,\lambda_{i-1}>\lambda_i}\left\langle\begin{matrix}\lambda+\varepsilon_i\\ \lambda\end{matrix}\right\rangle B'(\mu+\varepsilon_1,\lambda+\varepsilon_i)=B'(\mu,\lambda)-B'(\mu+\varepsilon_1,\lambda+\varepsilon_1).$$

The left-hand side equals

$$(-1)^{\lambda_1-\mu_1-1}\sum_{i\geqslant 2,\lambda_{i-1}>\lambda_i}\left\{\binom{\lambda_1-\mu_2}{\mu_1+1-\mu_2}\left\langle\begin{matrix}\mu'\\ \lambda'+\varepsilon_i\end{matrix}\right\rangle+\binom{\lambda_1-\mu_2-1}{\mu_1+1-\mu_2}\left\langle\begin{matrix}\mu''\\ \lambda'+\varepsilon_i\end{matrix}\right\rangle\right\}\left\langle\begin{matrix}\lambda+\varepsilon_i\\ \lambda\end{matrix}\right\rangle.$$

The right-hand side equals

$$\begin{aligned}(-1)^{\lambda_1-\mu_1}&\left\{\left(\binom{\lambda_1-\mu_2}{\mu_1-\mu_2}-\binom{\lambda_1+1-\mu_2}{\mu_1+1-\mu_2}\right)\left\langle\begin{matrix}\mu'\\ \lambda'\end{matrix}\right\rangle\right.\\ &\quad\left.+\left(\binom{\lambda_1-\mu_2-1}{\mu_1-\mu_2}-\binom{\lambda_1-\mu_2}{\mu_1+1-\mu_2}\right)\left\langle\begin{matrix}\mu''\\ \lambda'\end{matrix}\right\rangle\right\}\\ &=(-1)^{\lambda_1-\mu_1-1}\left(\binom{\lambda_1-\mu_2}{\mu_1+1-\mu_2}\left\langle\begin{matrix}\mu'\\ \lambda'\end{matrix}\right\rangle+\binom{\lambda_1-\mu_2-1}{\mu_1+1-\mu_2}\left\langle\begin{matrix}\mu''\\ \lambda'\end{matrix}\right\rangle\right).\end{aligned}$$

Similarly rewrite equation (2.4) as

$$\sum_{i\geqslant 2,\lambda_{i-1}>\lambda_i}\left\langle\begin{matrix}\lambda+\varepsilon_i\\ \lambda\end{matrix}\right\rangle B'(\sigma,\lambda+\varepsilon_i)=-B'(\sigma,\lambda+\varepsilon_1)$$

with the left-hand side (of course, $\sigma_1=\sigma_2$)

$$(-1)^{\lambda_1-\sigma_1}\sum_{i\geqslant 2,\lambda_{i-1}>\lambda_i}\left\{\binom{\lambda_1-\sigma_2}{\sigma_1-\sigma_2}\left\langle\begin{matrix}\sigma'\\ \lambda'+\varepsilon_i\end{matrix}\right\rangle+\binom{\lambda_1-\sigma_2-1}{\sigma_1-\sigma_2}\left\langle\begin{matrix}\sigma''\\ \lambda'+\varepsilon_i\end{matrix}\right\rangle\right\}\left\langle\begin{matrix}\lambda+\varepsilon_i\\ \lambda\end{matrix}\right\rangle$$

and the right-hand side

$$(-1)^{\lambda_1-\sigma_1}\left(\binom{\lambda_1-\sigma_2}{\sigma_1-\sigma_2}\left\langle\begin{matrix}\sigma'\\ \lambda'\end{matrix}\right\rangle+\binom{\lambda_1-\sigma_2-1}{\sigma_1-\sigma_2}\left\langle\begin{matrix}\sigma''\\ \lambda'\end{matrix}\right\rangle\right).$$

The goal is to show that the two sides are equal. We reduce the two cases to one by replacing $\sigma_1,\sigma',\sigma''$ by μ_1+1,μ',μ'', respectively. Let E denote the difference between the two sides (left $-$ right), and let

$$\begin{aligned}E_0&=-\binom{\lambda_1-\mu_2}{\mu_1+1-\mu_2}\left\langle\begin{matrix}\mu'\\ \lambda'\end{matrix}\right\rangle\\ E_1&=\binom{\lambda_1-\mu_2}{\mu_1+1-\mu_2}\sum_{i\geqslant 2,\lambda_{i-1}>\lambda_i}\left\langle\begin{matrix}\mu'\\ \lambda'+\varepsilon_i\end{matrix}\right\rangle\left\langle\begin{matrix}\lambda+\varepsilon_i\\ \lambda\end{matrix}\right\rangle-\binom{\lambda_1-\mu_2-1}{\mu_1+1-\mu_2}\left\langle\begin{matrix}\mu''\\ \lambda'\end{matrix}\right\rangle\\ E_2&=\binom{\lambda_1-\mu_2-1}{\mu_1+1-\mu_2}\sum_{i\geqslant 2,\lambda_{i-1}>\lambda_i}\left\langle\begin{matrix}\mu''\\ \lambda'+\varepsilon_i\end{matrix}\right\rangle\left\langle\begin{matrix}\lambda+\varepsilon_i\\ \lambda\end{matrix}\right\rangle\end{aligned}$$

then $E = (-1)^{\lambda_1-\mu_1-1}(E_0 + E_1 + E_2)$. We can assume $\mu' \supset \lambda'$ ($\mu \preceq \lambda$), because if any term in E is nonzero then $\left\langle {\mu' \atop \lambda'} \right\rangle \neq 0$ (if $\left\langle {\mu'' \atop \lambda'+\varepsilon_i} \right\rangle \neq 0$ for some i then $\mu' \supset \mu'' \supset \lambda' + \varepsilon_i \supset \lambda'$, and similarly if $\left\langle {\mu' \atop \lambda'+\varepsilon_i} \right\rangle \neq 0$ for some i then $\mu' \supset \lambda'$).

- *Case 1:* $|\mu'| = |\lambda'|$. Here $E_1 = 0 = E_2$, because any coefficient $\left\langle {\sigma \atop \tau} \right\rangle$ with $|\sigma| < |\tau|$ is zero. Further $\mu_1 = n - |\mu'| = \lambda_1$ and thus $E_0 = 0$ (from the binomial coefficient).
- *Case 2:* $|\mu'| = |\lambda'| + 1$ and $\mu_1 \geqslant \mu_2$. Here $E_2 = 0$ and $\lambda_1 = \mu_1 + 1$, thus the coefficient of $\left\langle {\mu'' \atop \lambda'} \right\rangle$ is zero (specifically, $\binom{\lambda_1-\mu_2-1}{\mu_1-\mu_2} - \binom{\lambda_1-\mu_2}{\mu_1+1-\mu_2} = 1 - 1$). The hypothesis implies $\mu' = \lambda' + \varepsilon_j$ for some $j \geqslant 2$, that is, $\mu = (\lambda_1 - 1, \lambda_2, \ldots, \lambda_j + 1, \ldots)$. If $j = 2$ then $\lambda_1 \geqslant \lambda_2 + 2$ otherwise $j > 2$ and $\lambda_1 - 1 \geqslant \lambda_{j-1} > \lambda_j$. Thus $E_0 + E_1 = \left\langle {\lambda+\varepsilon_j \atop \lambda} \right\rangle - \left\langle {\lambda'+\varepsilon_j \atop \lambda'} \right\rangle$; only the term with $i = j$ in the sum is nonzero (we showed $\lambda_{j-1} > \lambda_j$). By equation (2.1) $\left\langle {\lambda+\varepsilon_j \atop \lambda} \right\rangle = \left\langle {\lambda'+\varepsilon_j \atop \lambda'} \right\rangle$, since $1 \notin \{i : \lambda_i = \lambda_j + 1\}$.
- *Case 3:* $|\mu'| = |\lambda'| + 1$ and $\mu_1 = \mu_2 - 1$ (that is, $\mu = \sigma - \varepsilon_1$ where $\sigma_1 = \sigma_2$). Here $E_2 = 0$ and the binomial coefficients in E_0, E_1 are all equal to 1. Let $\mu' = \lambda' + \varepsilon_j$. If $\mu'' = \lambda'$ then either $j = 2, \lambda = (\lambda_1, \lambda_1 - 1, \lambda_3, \ldots), \mu = (\lambda_1 - 1, \lambda_1, \lambda_3, \ldots)$ with $\lambda_1 > \lambda_3$ or $j > 2, \lambda = (\lambda_1, \lambda_1, \ldots, \lambda_1, \lambda_1 - 1, \lambda_{j+1}, \ldots), \mu = (\lambda_1 - 1, \lambda_1, \ldots, \lambda_1, \lambda_{j+1}, \ldots)$ (that is, $\lambda_i = \lambda_1$ for $1 \leqslant i \leqslant j - 1$ and $\lambda_j = \lambda_1 - 1$), which implies

$$E_0 + E_1 = \left\langle {\mu' \atop \lambda' + \varepsilon_j} \right\rangle \left\langle {\lambda + \varepsilon_j \atop \lambda} \right\rangle - \left\langle {\mu'' \atop \lambda'} \right\rangle - \left\langle {\mu' \atop \lambda'} \right\rangle$$
$$= \left\langle {\lambda + \varepsilon_j \atop \lambda} \right\rangle - 1 - \left\langle {\lambda' + \varepsilon_j \atop \lambda'} \right\rangle = j - 1 - (j - 1)$$

  by equation (2.1). If $\mu'' \neq \lambda'$ then $\lambda_j + 1 < \lambda_2$ (so $j > 2$) and $E_0 + E_1 = \left\langle {\lambda+\varepsilon_j \atop \lambda} \right\rangle - \left\langle {\lambda'+\varepsilon_j \atop \lambda'} \right\rangle = 0$.
- *Case 4:* $|\mu'| \geqslant |\lambda'| + 2$ and $\mu_1 \geqslant \mu_2$. Thus $\lambda_1 \geqslant \mu_1 + 2 \geqslant \mu_2 + 2 \geqslant \lambda_2 + 2$ and equation (2.1) implies $\left\langle {\lambda+\varepsilon_i \atop \lambda} \right\rangle = \left\langle {\lambda'+\varepsilon_i \atop \lambda'} \right\rangle$ for each $i \geqslant 2, \lambda_{i-1} > \lambda_i$ (which includes $i = 2$). Also equation (2.2) applied to the truncated partition $\tau' = (\tau_2, \tau_3, \ldots)$ with $|\tau'| > |\lambda'|$ shows that

$$\sum \left\{ \left\langle {\tau' \atop \lambda' + \varepsilon_i} \right\rangle \left\langle {\lambda' + \varepsilon_i \atop \lambda'} \right\rangle : i = 2 \text{ or } i > 2, \lambda_{i-1} > \lambda \right\} = \left\langle {\tau' \atop \lambda'} \right\rangle.$$

  Substituting $\tau' = \mu'$ and $\tau' = \mu''$ in this equation shows that $E_0 + E_1 + E_2 = 0$.
- *Case 5:* $|\mu'| \geqslant |\lambda'| + 2$ and $\mu_1 = \mu_2 - 1$ (that is, $\mu = \sigma - \varepsilon_1$ where $\sigma_1 = \sigma_2$). Similarly $\lambda_1 \geqslant (\mu_2 - 1) + 2 \geqslant \lambda_2 + 1$, and $\left\langle {\lambda+\varepsilon_i \atop \lambda} \right\rangle = \left\langle {\lambda'+\varepsilon_i \atop \lambda'} \right\rangle$ for each $i > 2$ with $\lambda_{i-1} > \lambda_i$. Either $\lambda_1 \geqslant \lambda_2 + 2$ which implies $\left\langle {\lambda+\varepsilon_2 \atop \lambda} \right\rangle = \left\langle {\lambda'+\varepsilon_2 \atop \lambda'} \right\rangle = 1$ or $\lambda_1 = \lambda_2 + 1$ which implies $\mu_2 = \lambda_2$ and $\left\langle {\mu' \atop \lambda'+\varepsilon_2} \right\rangle = 0 = \left\langle {\mu'' \atop \lambda'+\varepsilon_2} \right\rangle$ because $(\lambda' + \varepsilon_2)_2 > \mu_2 \geqslant (\mu'')_2$. In both cases equation (2.2) shows $E_0 + E_1 + E_2 = 0$ as previously (in the case $\lambda_1 = \lambda_2 + 1$ one has $\left\langle {\lambda+\varepsilon_2 \atop \lambda} \right\rangle = 2, \left\langle {\lambda'+\varepsilon_2 \atop \lambda'} \right\rangle = 1$).

Thus the matrix $B'(\lambda, \mu)$ satisfies the hypotheses of the lemma, and the proof is completed. □

There was only one part of the proof which depended on the μ'' part of the formula: the case $|\mu'| = |\lambda'| + 1$, $\mu = \sigma - \varepsilon_1$ with $\sigma_1 = \sigma_2$ (of course the first steps of an induction proof are crucial!). By way of example, here is the matrix B for $\mathcal{P}_6^{(4)}$

$$\begin{bmatrix} 1 & -2 & 0 & -2 & 8 & 0 & 2 & -18 & 18 \\ 0 & 1 & 0 & 0 & -10 & 0 & 0 & 42 & -72 \\ 0 & 0 & 1 & -3 & 3 & 0 & 3 & -9 & 9 \\ 0 & 0 & 0 & 1 & -3 & 0 & -2 & 13 & -18 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -9 & 24 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

with the rows and columns labelled by [2211, 3111, 222, 321, 411, 33, 42, 51, 6] (suppressing the commas in the partitions). We point out an application of the $\{\widetilde{m}_\lambda\}$-basis: suppose that a symmetric polynomial is to be restricted to the subspace $\left\{x : \sum_{i=1}^N x_i = 0\right\}$; this can be done by expressing it in terms of $\widetilde{m}_\lambda$ (using the coefficients $B(\lambda, \mu)$) and then restricting to the subspace, with the effect of annihilating all $\widetilde{m}_\lambda$ with $\lambda_1 > \lambda_2$ and leaving an expression in the basis $\{m_\lambda : \lambda_1 = \lambda_2\}$.

3. Invariant harmonic polynomials

In the following, the symmetric polynomials have the argument $x^2 = \left(x_1^2, \ldots, x_N^2\right)$ (and y^2 is similarly defined). Thus $e_1(x^2) = \|x\|^2$, and the former will often be used in equations involving symmetric functions. We recall the basic facts about the Poisson kernel, that is, the reproducing kernel, for harmonic polynomials $\mathbb{H}_n$, (see Dunkl and Xu [5, ch 5]). The intertwining operator V is a linear isomorphism on polynomials such that $V\mathbb{P}_n^{(N)} = \mathbb{P}_n^{(N)}$ for each n, $\mathcal{D}_i V = V\frac{\partial}{\partial x_i}$ for $1 \leqslant i \leqslant N$, and $V1 = 1$. Here the reflection group is a direct product and so the intertwining map is the N-fold tensor product of the one-dimensional transform (see [3, theorem 5.1]), defined by (for $n \geqslant 0$)

$$Vx_1^{2n} = \frac{\left(\frac{1}{2}\right)_n}{\left(\kappa + \frac{1}{2}\right)_n} x_1^{2n} \qquad Vx_1^{2n+1} = \frac{\left(\frac{1}{2}\right)_{n+1}}{\left(\kappa + \frac{1}{2}\right)_{n+1}} x_1^{2n+1}.$$

Then let $K_n(x, y) = \frac{1}{n!} V^{(x)}(\langle x, y\rangle^n)$, for $x, y \in \mathbb{R}^N$, $n \geqslant 0$ ($V^{(x)}$ acts on x). The key properties of K_n needed here are $K_n(xw, yw) = K_n(x, y)$ for $w \in B_N$ and $\mathcal{D}_i^{(x)} K_n(x, y) = y_i K_{n-1}(x, y)$ for $1 \leqslant i \leqslant N$.

Definition 7. *Let* $\mathrm{d}\omega$ *denote the normalized rotation-invariant surface measure on the unit sphere* $S = \{x \in \mathbb{R}^N : \|x\| = 1\}$, *and for polynomials* f, g *the inner product is*

$$\langle f, g\rangle_S = c_\kappa \int_S f(x) g(x) \prod_{i=1}^N |x_i|^{2\kappa}\, \mathrm{d}\omega$$

where $c_\kappa = \dfrac{\Gamma\left(\frac{N}{2} + N\kappa\right)\Gamma\left(\frac{1}{2}\right)^N}{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\kappa + \frac{1}{2}\right)^N}$ *so that* $\langle 1, 1\rangle_S = 1$.

The reproducing kernel for $\mathbb{H}_n$, $n \geqslant 1$ is

$$P_n(x, y) = \left(\frac{N}{2} + N\kappa + n - 1\right) \sum_{j \leqslant n/2} (-1)^j \frac{\left(\frac{N}{2} + N\kappa\right)_{n-1-j}}{j!} 2^{n-2j} \|x\|^{2j} \|y\|^{2j} K_{n-2j}(x, y)$$

and satisfies $\langle P_n(\cdot, y), f\rangle_S = f(y)$ for each $f \in \mathbb{H}_n$. The general theory [2] for our differential-difference operators shows that $f \in \mathbb{H}_n, g \in \mathbb{H}_m, m \neq n$ imply $\langle f, g\rangle_S = 0$.

Let K^0_{2n} denote the kernel $K_{2n}(x,y)$ symmetrized with respect to the group $\mathbb{Z}_2^N$ (for fixed $x \in \mathbb{R}^N$ sum over the 2^N points $(\pm x_1, \ldots, \pm x_N)$ and divide by 2^N); it is the reproducing kernel for $\mathbb{H}^0_{2n}$. Let $K^B_{2n}(x,y)$, $P^B_{2n}(x,y)$ denote the symmetrizations of $K_{2n}(x,y)$, $P_{2n}(x,y)$ respectively, with respect to the group B_N. Thus $K^B_{2n}(x,y) = \frac{1}{N!}\sum_{w\in S_N} K^0_{2n}(xw,y)$.

Proposition 5. *For $n \geqslant 1$, $x, y \in \mathbb{R}^N$*

$$K^0_{2n}(x,y) = 2^{-2n} \sum_{\alpha\in\mathbb{N}_0^N, |\alpha|=n} \frac{1}{\alpha!\left(\kappa+\frac{1}{2}\right)_\alpha} x^{2\alpha} y^{2\alpha}$$

$$K^B_{2n}(x,y) = 2^{-2n} \sum_{\lambda\in\mathcal{P}_n^{(N)}} \frac{1}{\lambda!\left(\kappa+\frac{1}{2}\right)_\lambda} \frac{m_\lambda(x^2)m_\lambda(y^2)}{m_\lambda(1^N)}.$$

Proof. By the multinomial theorem, $\frac{\langle x,y\rangle^{2n}}{(2n)!} = \frac{1}{(2n)!}\sum\left\{\binom{2n}{\beta}x^\beta y^\beta : \beta \in \mathbb{N}_0^N, |\beta| = 2n\right\}$. Symmetrizing over $\mathbb{Z}_2^N$ removes monomials with odd exponents, and thus $K^0_{2n}(x,y)$ is the result of applying V to $\sum\left\{\frac{1}{(2\alpha)!}x^{2\alpha}y^{2\alpha} : \alpha \in \mathbb{N}_0^N, |\alpha| = n\right\}$. Further $(2\alpha)! = 2^{2|\alpha|}\alpha!\left(\frac{1}{2}\right)_\alpha$, and $Vx^{2\alpha} = \left(\left(\frac{1}{2}\right)_\alpha\big/\left(\kappa+\frac{1}{2}\right)_\alpha\right)x^{2\alpha}$. For a fixed $\lambda \in \mathcal{P}_n^{(N)}$ the sum $\frac{1}{N!}\sum\{x^{2w\alpha}y^{2\alpha} : w \in S_N, \alpha^+ = \lambda\}$ is a multiple of $m_\lambda(x^2)m_\lambda(y^2)$, and evaluated at $x = 1^N = y$ the sum equals $m_\lambda(1^N)$ (that is, $\#\left\{\alpha \in \mathbb{N}_0^N : \alpha^+ = \lambda\right\}$). The terms $\alpha!$ and $\left(\kappa+\frac{1}{2}\right)_\alpha$ are invariant under S_N. □

Corollary 1. *For $n \geqslant 1$, $x, y \in \mathbb{R}^N$*

$$P^B_{2n}(x,y) = \left(\frac{N}{2}+N\kappa+2n-1\right)\sum_{j=0}^{n}(-1)^j \frac{\left(\frac{N}{2}+N\kappa\right)_{2n-1-j}}{j!} e_1(x^2)^j e_1(y^2)^j \times \sum_{\lambda\in\mathcal{P}_{n-j}^{(N)}} \frac{1}{\lambda!\left(\kappa+\frac{1}{2}\right)_\lambda}\frac{m_\lambda(x^2)m_\lambda(y^2)}{m_\lambda(1^N)}.$$

Let $\mathbb{P}^B_{2n}$ denote the space of B_N-invariant elements of $\mathbb{P}^{(N)}_{2n}$, then $\dim \mathbb{P}^B_{2n} = \#\mathcal{P}_n^{(N)}$. The space of B_N-invariant harmonic polynomials $\mathbb{H}^B_{2n}$ is the kernel of Δ_κ; since Δ_κ commutes with the action of B_N and maps $\mathbb{P}^{(N)}_{2n}$ onto $\mathbb{P}^{(N)}_{2n-2}$ we see that $\dim \mathbb{H}^B_{2n} = \#\mathcal{P}_n^{(N)} - \#\mathcal{P}_{n-1}^{(N)}$. The map $\lambda \mapsto \lambda + \varepsilon_1$ is a one-to-one correspondence of $\mathcal{P}_{n-1}^{(N)}$ onto a subset of $\mathcal{P}_n^{(N)}$ whose complement is

$$\widetilde{\mathcal{P}}_n^{(N)} = \left\{\lambda \in \mathcal{P}_n^{(N)} : \lambda_1 = \lambda_2\right\}.$$

Thus $\dim \mathbb{H}^B_{2n} = \#\widetilde{\mathcal{P}}_n^{(N)}$, and we will construct a basis for $\mathbb{H}^B_{2n}$ whose elements are labelled in a natural way by $\widetilde{\mathcal{P}}_n^{(N)}$. There is a generating function for the dimensions: for $\lambda \in \widetilde{\mathcal{P}}_n^{(N)}$ the conjugate λ^T (the partition corresponding to the transpose of the Ferrers diagram) of λ is a partition with $2 \leqslant (\lambda^T)_i \leqslant N$ for all i (no parts equal to 1 or exceeding N); thus

$$\sum_{n=0}^{\infty}\left(\#\widetilde{\mathcal{P}}_n^{(N)}\right)q^n = \prod_{j=2}^{N}(1-q^j)^{-1}.$$

This expression yields an estimate for $\#\widetilde{\mathcal{P}}_n^{(N)}$. Indeed, let $M = \operatorname{lcm}(2,3,\ldots,N)$, $\prod_{j=2}^N(1-q^j)^{-1} = p(q)(1-q^M)^{-(N-1)}$ for some polynomial $p(q)$; thus $\#\widetilde{\mathcal{P}}_n^{(N)} = O\left(\left(\frac{n}{M}\right)^{N-2}\right)$ as $n \to \infty$. In $K^B_{2n}(x,y)$ expand each $m_\lambda(y^2)$ in the $\widetilde{m}_\mu$-basis

$$K^B_{2n}(x,y) = 2^{-2n}\sum_{\mu\preceq\lambda}\frac{B(\mu,\lambda)}{\lambda!\left(\kappa+\frac{1}{2}\right)_\lambda}\frac{m_\lambda(x^2)}{m_\lambda(1^N)}\widetilde{m}_\mu(y^2).$$

This leads to the following.

Definition 8. *For $\mu \in \mathcal{P}_n^{(N)}$ the B_N-invariant polynomial h_μ is given by*

$$h_\mu(x) = 2^{-n} \sum_{\lambda \in \mathcal{P}_n^{(N)}, \mu \preceq \lambda} \frac{B(\mu, \lambda)}{\lambda!\left(\kappa + \frac{1}{2}\right)_\lambda} \frac{m_\lambda(x^2)}{m_\lambda(1^N)}.$$

Recall that $\lambda \in \mathcal{P}_n$ and $B(\lambda, \mu) \neq 0$ imply $l(\lambda) \leqslant l(\mu)$ and $\lambda \in \mathcal{P}_n^{(N)}$.

Theorem 2. *For $\mu \in \mathcal{P}_n^{(N)}$, if $\mu_1 > \mu_2$ then $\Delta_\kappa h_\mu = 2h_{\mu-\varepsilon_1}$ and if $\mu_1 = \mu_2$ then $\Delta_\kappa h_\mu = 0$.*

Proof. From the basic properties of K_{2n} it follows that $\Delta_\kappa^{(x)} K_{2n}(x, y) = \|y\|^2 K_{2n-2}(x, y)$. Symmetrize this equation with respect to B_N (and note that Δ_κ commutes with the group action) to obtain

$$\begin{aligned} \Delta_\kappa^{(x)} K_{2n}^B(x, y) &= 2^{-n} \sum_{\mu \in \mathcal{P}_n^{(N)}} \Delta_\kappa^{(x)} h_\mu(x) \widetilde{m}_\mu(y^2) \\ &= e_1(y^2) K_{2n-2}^B(x, y) \\ &= 2^{1-n} \sum_{\sigma \in \mathcal{P}_{n-1}^{(N)}} h_\sigma(x) \widetilde{m}_{\sigma+\varepsilon_1}(y^2). \end{aligned}$$

By definition $e_1(y^2)\widetilde{m}_\sigma(y^2) = \widetilde{m}_{\sigma+\varepsilon_1}(y^2)$. Considering the equations as expansions in $\{\widetilde{m}_\mu(y^2) : \mu \in \mathcal{P}_n^{(N)}\}$ shows $\Delta_\kappa h_{\sigma+\varepsilon_1}(x) = 2h_\sigma(x)$ for $\sigma \in \mathcal{P}_{n-1}^{(N)}$ and $\Delta_\kappa h_\mu(x) = 0$ if $\mu_1 = \mu_2$. □

Corollary 2. *The set $\left\{h_\mu : \mu \in \widetilde{\mathcal{P}}_n^{(N)}\right\}$ is a basis for $\mathbb{H}_{2n}^B$.*

Proof. For $\lambda, \mu \in \widetilde{\mathcal{P}}_n^{(N)}$ the coefficient of m_λ in h_μ is $\delta_{\lambda\mu}\left(2^n\lambda!\left(\kappa + \frac{1}{2}\right)_\lambda m_\lambda(1^N)\right)^{-1}$ and thus $\left\{h_\mu : \mu \in \widetilde{\mathcal{P}}_n^{(N)}\right\}$ is linearly independent. Also $\dim \mathbb{H}_{2n}^B = \#\widetilde{\mathcal{P}}_n^{(N)}$. □

Besides the inner product on polynomials defined by integration over the sphere S there is also the important pairing defined in an algebraic manner using the operators $\mathcal{D}_i$, namely

$$\langle f, g\rangle_h = f(\mathcal{D}_1, \ldots, \mathcal{D}_N)\, g(x)|_{x=0}.$$

Since $\mathcal{D}_i^2 x_i^{2n} = 2n(2n - 1 + 2\kappa)x_i^{2n-2}$ and $\mathcal{D}_j^2 x_i^{2n} = 0$ for $j \neq i$, we have that $\left\langle x_i^{2n}, x_i^{2n}\right\rangle_h = 2^{2n} n!\left(\kappa + \frac{1}{2}\right)_n$ and $\langle x^{2\alpha}, x^{2\beta}\rangle_h = \delta_{\alpha\beta} 2^{2|\alpha|}\alpha!\left(\kappa + \frac{1}{2}\right)_\alpha$ for $\alpha, \beta \in \mathbb{N}_0^N$, so that monomials are mutually orthogonal. It follows that $\langle m_\lambda(x^2), m_\mu(x^2)\rangle_h = \delta_{\lambda\mu} 2^{2n}\lambda!\left(\kappa + \frac{1}{2}\right)_\lambda m_\lambda(1^N)$ for $\lambda, \mu \in \mathcal{P}_n^{(N)}$. It was shown in ([3, theorem 3.8], see also [5, theorem 5.2.4]) that for $f, g \in \mathbb{H}_n$

$$\langle f, g\rangle_h = 2^n \left(\frac{N}{2} + N\kappa\right)_n \langle f, g\rangle_S\,.$$

This allows the direct calculation of $\langle h_\lambda, h_\mu\rangle_S$ for $\lambda, \mu \in \widetilde{\mathcal{P}}_n^{(N)}$.

Proposition 6. *For $\lambda, \mu \in \widetilde{\mathcal{P}}_n^{(N)}$*

$$\langle h_\lambda, h_\mu\rangle_h = \sum_{\lambda \preceq \sigma, \mu \preceq \sigma} \frac{B(\lambda, \sigma) B(\mu, \sigma)}{\sigma!\left(\kappa + \frac{1}{2}\right)_\sigma m_\sigma(1^N)}$$

$$\langle h_\lambda, h_\mu\rangle_S = 2^{-2n}\left(\left(\frac{N}{2} + N\kappa\right)_{2n}\right)^{-1} \langle h_\lambda, h_\mu\rangle_h.$$

Computations with small degrees ($n \leqslant 8$) show that there are no product (linear factors in κ) formulae for the inner products and $\{h_\lambda\}$ is not an orthogonal basis. But the bi-orthogonal set for $\left\{h_\mu : \mu \in \widetilde{\mathcal{P}}_n^{(N)}\right\}$ can be described exactly. The idea is to extract a certain multiple of the coefficient of $m_\lambda(y^2)$ for $\lambda \in \widetilde{\mathcal{P}}_n^{(N)}$ in the Poisson kernel $P_{2n}^B(x, y)$. Indeed P_{2n}^B is a multiple of

$$\sum_{j=0}^{n} \sum_{\sigma \in \mathcal{P}_{n-j}^{(N)}} (-1)^j\, e_1(x^2)^j e_1(y^2)^j \frac{\left(\frac{N}{2} + N\kappa\right)_{2n-1-j}}{j!\sigma!\left(\kappa + \frac{1}{2}\right)_\sigma} \frac{m_\sigma(x^2) m_\sigma(y^2)}{m_\sigma(1^N)}$$

$$= \sum_{j=0}^{n} \sum_{\lambda \in \mathcal{P}_n^{(N)}} \sum_{\sigma \in \mathcal{P}_{n-j}^{(N)}} (-1)^j\, e_1(x^2)^j \left\langle \begin{matrix} \lambda \\ \sigma \end{matrix} \right\rangle \frac{\left(\frac{N}{2} + N\kappa\right)_{2n-1-j}}{j!\sigma!\left(\kappa + \frac{1}{2}\right)_\sigma} \frac{m_\sigma(x^2) m_\lambda(y^2)}{m_\sigma(1^N)}.$$

Then let

$$g_\lambda(x) = \sum_{j=0}^{n} \frac{e_1(x^2)^j}{j!\left(-\frac{N}{2} - N\kappa - 2n + 2\right)_j} \sum_{\sigma \in \mathcal{P}_{n-j}^{(N)}, \sigma \subset \lambda} \left\langle \begin{matrix} \lambda \\ \sigma \end{matrix} \right\rangle \frac{1}{\sigma!\left(\kappa + \frac{1}{2}\right)_\sigma} \frac{m_\sigma(x^2)}{m_\sigma(1^N)}$$

so that the coefficient of $m_\lambda(y^2)$ in $P_{2n}^B(x, y)$ is $\left(\frac{N}{2} + N\kappa\right)_{2n} g_\lambda(x)$. Observe that $g_\lambda(x) = \frac{1}{\lambda!\left(\kappa + \frac{1}{2}\right)_\lambda} \frac{m_\lambda(x^2)}{m_\lambda(1^N)} + \|x\|^2 g'_\lambda(x)$ where $g'_\lambda(x)$ is a polynomial of degree $2n - 2$.

Proposition 7. *For $\lambda, \mu \in \widetilde{\mathcal{P}}_n^{(N)}$ the inner product*

$$\langle g_\lambda, h_\mu \rangle_h = \delta_{\lambda\mu} \frac{2^n}{\lambda!\left(\kappa + \frac{1}{2}\right)_\lambda m_\lambda(1^N)}.$$

Proof. Indeed

$$\langle g_\lambda, h_\mu \rangle_h = \frac{1}{\lambda!\left(\kappa + \frac{1}{2}\right)_\lambda m_\lambda(1^N)} \langle m_\lambda(x^2), h_\mu(x) \rangle_h + g'_\lambda(\mathcal{D}) \Delta_\kappa h_\mu(x)$$

$$= \delta_{\lambda\mu} \frac{2^n}{\lambda!\left(\kappa + \frac{1}{2}\right)_\lambda m_\lambda(1^N)}$$

by the properties of the pairing $\langle \cdot, \cdot \rangle_h$. □

Essentially g_λ is (a scalar multiple of) the projection of $m_\lambda(x^2)$ onto $\mathbb{H}_{2n}$. The method of orthogonal projections to construct bases of harmonic polynomials was studied in more detail by Xu [10].

Although the formulae for the inner products of the h_λ are complicated, the determinant of the Gram matrix $\left(\langle h_\lambda, h_\mu \rangle_h\right)_{\lambda,\mu \in \widetilde{\mathcal{P}}_n^{(N)}}$ has an elegant expression in terms of linear factors in the parameter κ. One reason that this matters is that the singularities (both zeros and poles) in the L^2-norms of classical orthogonal polynomials are closely related to the underlying algebraic structure. That is, the norms involve gamma functions (for example, Jacobi polynomials) and the poles are determined by linear functions of the parameters (with rational coefficients). In the present situation there is a parametrized operator and inner-product structure and the formula for the Gram determinant precisely exhibits the singularities. We now evaluate the determinant of the Gram matrix of $\left\{h_\lambda : \lambda \in \widetilde{\mathcal{P}}_n^{(N)}\right\}$ using the $\langle \cdot, \cdot \rangle_h$ inner product; the value for $\langle \cdot, \cdot \rangle_S$ is just a product of a power $\left(\#\widetilde{\mathcal{P}}_n^{(N)}\right)$ of the proportionality factor with the previous one. It is perhaps surprising that the calculation can be carried out by finding the determinant for the entire set $\left\{h_\lambda : \lambda \in \mathcal{P}_n^{(N)}\right\}$, which can easily be done. By the orthogonality of $\{m_\mu\}$ we see that the Gram matrix $G(\lambda, \mu) = \langle h_\lambda, h_\mu \rangle_h$ for $\lambda, \mu \in \mathcal{P}_n^{(N)}$ satisfies

$$G = BCB^T \qquad \det G = (\det B)^2 \det C = \det C$$

where the diagonal matrix $C(\lambda,\mu) = \delta_{\lambda\mu}\left(\lambda!\left(\kappa+\frac{1}{2}\right)_\lambda m_\lambda(1^N)\right)^{-1}$; just as in proposition 6 $\langle h_\lambda, h_\mu\rangle_h = \sum_{\lambda\preceq\sigma,\mu\preceq\sigma} \frac{B(\lambda,\sigma)B(\mu,\sigma)}{\sigma!\left(\kappa+\frac{1}{2}\right)_\sigma m_\sigma(1^N)}$ for any $\lambda,\mu\in\mathcal{P}_n^{(N)}$. We already showed that B is triangular with 1s on the main diagonal so that $\det B = 1$. Let

$$D_n = \det C = \prod_{\lambda\in\mathcal{P}_n^{(N)}} \left(\lambda!\left(\kappa+\frac{1}{2}\right)_\lambda m_\lambda(1^N)\right)^{-1}$$

and let $\widetilde{D}_n = \det\widetilde{G}$, where $\widetilde{G}$ is the principal submatrix of G for the labels $\lambda,\mu\in\widetilde{\mathcal{P}}_n^{(N)}$. We will show that

$$\widetilde{D}_n = \frac{D_n}{D_{n-1}} \prod_{\lambda\in\mathcal{P}_{n-1}^{(N)}} \left((\lambda_1-\lambda_2+1)\left(N\kappa+\frac{N}{2}+2(n-1)-(\lambda_1-\lambda_2)\right)\right).$$

A simplified form of this will be given later. The idea of the proof is to use the orthogonal decomposition $\mathbb{P}_n^{(N)} = \bigoplus_{j\leqslant n/2}\|x\|^{2j}\mathbb{H}_{n-2j}$ to produce a transformation of the Gram matrix into block form. The underlying relation is the product formula for Δ_κ: for $m, j\geqslant 0$ and any $f\in\mathbb{P}_m^{(N)}$

$$\Delta_\kappa\|x\|^{2j}f(x) = 4j\left(m+j-1+\frac{N}{2}+N\kappa\right)\|x\|^{2j-2}f(x) + \|x\|^{2j}\Delta_\kappa f(x).$$

Specializing to harmonic polynomials $f\in\mathbb{H}_m$ and iterating this formula shows that

$$\Delta_\kappa^s\|x\|^{2j}f(x) = 2^{2s}(-j)_s\left(-m-j+1-\frac{N}{2}-N\kappa\right)_s\|x\|^{2j-2s}f(x) \tag{3.1}$$

for $s\geqslant 0$; note $\Delta_\kappa^s\|x\|^{2j}f(x) = 0$ for $s > j$ (see [3, theorem 3.6]). Suppose $f\in\mathbb{H}_{n-2i}$, $g\in\mathbb{H}_{n-2j}$ $\left(i,j\leqslant\frac{n}{2}\right)$ then

$$\langle\|x\|^{2i}f(x), \|x\|^{2j}g(x)\rangle_h = \delta_{ij}2^{2i}i!\left(n-2i+\frac{N}{2}+N\kappa\right)_i\langle f,g\rangle_h$$

(by the symmetry of the inner product we can assume $i > j$ and use equation (3.1)).

Lemma 2. *Suppose $f\in\mathbb{P}_n^{(N)}$ and $f(x) = \sum_{j\leqslant n/2}\|x\|^{2j}f_{n-2j}(x)$ with $f_{n-2j}\in\mathbb{H}_{n-2j}$ for each $j\leqslant\frac{n}{2}$ then $\Delta_\kappa^i f = 2^{2i}i!\left(n-2i+\frac{N}{2}+N\kappa\right)_i f_{n-2i}$ if and only if $f_{n-2j}=0$ for each $j>i$, (that is, $\Delta_\kappa^{i+1}f = 0$).*

Proof. By equation (3.1) and for each $i\leqslant\frac{n}{2}$

$$\Delta_\kappa^i f(x) = \sum_{i\leqslant j\leqslant n/2} 2^{2i}(-j)_i\left(-n+j+1-\frac{N}{2}-N\kappa\right)_i\|x\|^{2j-2i}f_{n-2j}(x).$$

This is an orthogonal expansion, hence $\Delta_\kappa^i f(x)$ equals the term in the sum with $j = i$ if and only if $f_{n-2j} = 0$ for all $j > i$. □

The lemma allows us to find the lowest-degree part of the expansion of h_μ for $\mu\in\mathcal{P}_n^{(N)}$, since $\Delta_\kappa^i h_\mu = 2^i h_{\mu-i\varepsilon_1}$ if $i\leqslant\mu_1-\mu_2$ and $\Delta_\kappa^i h_\mu = 0$ if $i>\mu_1-\mu_2$, by theorem 2. We will use an elementary property of any inner-product space E: suppose $\{f_i : 1\leqslant i\leqslant m\}$ is a linearly independent set in E, for some $k<m$ let ρ denote the orthogonal projection of E onto $\operatorname{span}\{f_i : 1\leqslant i\leqslant k\}$, then

$$\det(\langle f_i, f_j\rangle)_{i,j=1}^m = \det(\langle f_i, f_j\rangle)_{i,j=1}^k \det(\langle(1-\rho)f_i, (1-\rho)f_j\rangle)_{i,j=k+1}^m.$$

The proof is easy: for each $i > k$ there are coefficients a_{ji} for $1 \leqslant j \leqslant m$ so that $\rho f_i = \sum_{j=1}^{k} a_{ji} f_j$. Let A be the $m \times m$ matrix with entries $A_{ji} = \delta_{ji}$ except $A_{ji} = -a_{ji}$ for $1 \leqslant j \leqslant k < i \leqslant m$. Let G be the Gram matrix of $\{f_i\}_{i=1}^m$ and let $G' = A^T G A$ so that

$$G'_{ij} = G'_{ji} = \begin{cases} \langle f_i, f_j\rangle & \text{for} \quad 1 \leqslant i \leqslant j \leqslant k \\ \langle f_i, (1-\rho) f_j\rangle = 0 & \text{for} \quad 1 \leqslant i \leqslant k < j \leqslant m \\ \langle (1-\rho) f_i, (1-\rho) f_j\rangle & \text{for} \quad k+1 \leqslant i \leqslant j \leqslant m. \end{cases}$$

Thus $\det G' = (\det A)^2 \det G = \det G$ and G' has a 2×2 block structure with 0 in the off-diagonal blocks. We apply this to the projection of $\mathbb{P}_{2n}^B$ onto $\mathbb{H}_{2n}^B$, denoted by ρ_n. We write e_1 for $\|x\|^2$ as before. For $\mu \in \mathcal{P}_n^{(N)}$ let $h_\mu = \sum_{j=0}^{\mu_1-\mu_2} e_1^j h_{\mu,j}(x)$ with each $h_{\mu,j} \in \mathbb{H}_{2n-2j}^B$ (and by the above discussion, $h_{\mu,i}$ is a multiple of $h_{\mu - i\varepsilon_1}$ for $i = \mu_1 - \mu_2$), then $(1-\rho_n)h_\mu = \sum_{j=1}^{\mu_1-\mu_2} e_1^j h_{\mu,j}$. Thus $D_n = \widetilde{D}_n \det M$ where M is the Gram matrix for $\left\{(1-\rho_n)h_\mu : \mu \in \mathcal{P}_n^{(N)}, \mu_1 > \mu_2\right\}$. The span of this set is $e_1 \mathbb{P}_{2n-2}^B$, and $\mathbb{P}_{2n-2}^B$ can be decomposed just like $\mathbb{P}_{2n}^B$. Indeed $\left\{(1-\rho_n)h_\mu : \mu \in \mathcal{P}_n^{(N)}, \mu_1 = \mu_2 + 1\right\}$ is a basis for $e_1 \mathbb{H}_{2n-2}^B$, since $(1-\rho_n)h_\mu = e_1 h_{\mu,1}$, a nonzero multiple of $e_1 h_{\mu-\varepsilon_1}$, for $\mu_1 = \mu_2 + 1$. Repeat the previous step with the projection $e_1 \rho_{n-1} e_1^{-1}$ to express $\det M$ as a product. Here is a formal inductive argument.

Let $M^{(j)}(\lambda, \mu)$ be the symmetric matrix indexed by $\mathcal{P}_n^{(N)}$ with entries

$$M^{(j)}(\lambda,\mu) = \begin{cases} \left\langle e_1^i h_{\lambda,i}, e_1^i h_{\mu,i}\right\rangle_h & \text{for} \quad \lambda_1 - \lambda_2 = \mu_1 - \mu_2 = i < j \\ 0 & \text{for} \quad \lambda_1 - \lambda_2 < \min(\mu_1 - \mu_2, j) \\ \left\langle \sum_{i=j}^{\lambda_1-\lambda_2} e_1^i h_{\lambda,i}, \sum_{i=j}^{\mu_1-\mu_2} e_1^i h_{\mu,i} \right\rangle_h & \text{for} \quad \lambda_1 - \lambda_2, \mu_1 - \mu_2 \geqslant j. \end{cases}$$

By the symmetry $M^{(j)}(\lambda,\mu) = 0$ if $\mu_1 - \mu_2 < \min(\lambda_1 - \lambda_2, j)$. The matrix $M^{(N)}$ has a diagonal block decomposition (zero blocks off the diagonal) with one block for each set of labels $\left\{\lambda \in \mathcal{P}_n^{(N)} : \lambda_1 = \lambda_2 + i\right\}$, equivalently $\widetilde{\mathcal{P}}_{n-i}^{(N)}$, for each $i = 0, 1, \ldots, n-2, n$ (the set $\widetilde{\mathcal{P}}_1^{(N)}$ is empty). The matrix $M^{(0)} = G$, the Gram matrix of $\left\{h_\lambda : \lambda \in \mathcal{P}_n^{(N)}\right\}$. We show that $\det G = \det M^{(0)}$ by proving $\det M^{(j)} = \det M^{(j+1)}$ for each $j < n$. Only the principal submatrix of $M^{(j)}$ labelled by λ with $\lambda_1 - \lambda_2 \geqslant j$ need be considered. This is the Gram matrix of a certain basis for $e_1^j \mathbb{P}_{2n-2j}^B$; the projection $e_1^j \rho_{n-j} e_1^{-j}$ maps this space onto $e_1^j \mathbb{H}_{2n-2j}^B = \operatorname{span}\left\{e_1^j h_{\lambda,j} : \lambda_1 = \lambda_2 + j\right\}$. Now observe that $\left(1 - e_1^j \rho_{n-j} e_1^{-j}\right) \sum_{i=j}^{\lambda_1-\lambda_2} e_1^i h_{\lambda,i} = \sum_{i=j+1}^{\lambda_1-\lambda_2} e_1^i h_{\lambda,i}$ for λ with $\lambda_1 - \lambda_2 > j$. The projection argument shows $\det M^{(j)} = \det M^{(j+1)}$.

The determinant of the principal submatrix of $M^{(N)}$ labelled by $\left\{\lambda \in \mathcal{P}_n^{(N)} : \lambda_1 = \lambda_2 + i\right\}$ is a multiple of $\widetilde{D}_{n-i}$. Let

$$c_i = \left(2^{2i} i! \left(2n - 2i + \frac{N}{2} + N\kappa\right)_i\right)^{-1}.$$

By lemma 2 $h_{\lambda,i} = c_i \Delta_\kappa^i h_\lambda = 2^i c_i h_{\lambda - i\varepsilon_1}$ for $\lambda_1 = \lambda_2 + i$. Further

$$\left\langle e_1^i h_{\lambda,i}, e_1^i h_{\mu,i}\right\rangle_h = c_i^{-1} \langle h_{\lambda,i}, h_{\mu,i}\rangle_h = 2^{2i} c_i \left\langle h_{\lambda - i\varepsilon_1}, h_{\mu - i\varepsilon_1}\right\rangle_h$$

for $\lambda_1 - \lambda_2 = \mu_1 - \mu_2 = i$. The correspondence $\sigma \mapsto \sigma + i\varepsilon_1$ is one-to-one from $\widetilde{\mathcal{P}}_{n-i}^{(N)}$ to $\left\{\lambda \in \mathcal{P}_n^{(N)} : \lambda_1 = \lambda_2 + i\right\}$. Thus

$$D_n = \det G = \prod_{i=0}^{n} \left(i! \left(2n - 2i + \frac{N}{2} + N\kappa\right)_i\right)^{-d(n-i,N)} \widetilde{D}_{n-i}$$

where $d(j, N) = \#\widetilde{\mathcal{P}}_j^{(N)}$, for $j \geqslant 0$. In particular, $d(1, N) = 0$ and $\widetilde{D}_1 = 1$.

Theorem 3. *The determinant $\widetilde{D}_n$ of the Gram matrix of $\{h_\lambda : \lambda \in \widetilde{\mathcal{P}}_n^{(N)}\}$ for the inner product $\langle\cdot,\cdot\rangle_h$ satisfies*

$$\widetilde{D}_n = \frac{D_n}{D_{n-1}} \prod_{i=1}^{n} \left(i \left(2n - i - 1 + \frac{N}{2} + N\kappa \right) \right)^{d(n-i,N)}$$

and for $n \geqslant 2$ the following holds:

$$\widetilde{D}_n = \prod_{\lambda\in\widetilde{\mathcal{P}}_n^{(N)}} \left(\lambda! \left(\kappa + \frac{1}{2} \right)_\lambda m_\lambda(1^N) \right)^{-1} \times \prod_{\mu\in\mathcal{P}_{n-1}^{(N)}} \frac{(\mu_1 - \mu_2 + 1)\left(2n - 2 - \mu_1 + \mu_2 + \frac{N}{2} + N\kappa\right)}{(\mu_1 + 1)\left(\kappa + \frac{1}{2} + \mu_1\right)\#\{j : \mu_j = \mu_1\}}.$$

Proof. From the above result

$$\frac{D_n}{D_{n-1}} = \widetilde{D}_n \frac{\prod_{j=0}^{n-1}\left(j!\left(2n-2-2j+\frac{N}{2}+N\kappa\right)_j\right)^{d(n-1-j,N)}}{\prod_{i=1}^{n}\left(i!\left(2n-2i+\frac{N}{2}+N\kappa\right)_i\right)^{d(n-i,N)}}.$$

In the numerator replace j by $i-1$; the ratio $\frac{\left(2n-2i+\frac{N}{2}+N\kappa\right)_{i-1}}{\left(2n-2i+\frac{N}{2}+N\kappa\right)_i} = \frac{1}{2n-i-1+\frac{N}{2}+N\kappa}$, and this proves the first formula. The ratio $\frac{D_n}{D_{n-1}}$ can be simplified by using decomposition $\mathcal{P}_n^{(N)} = \widetilde{\mathcal{P}}_n^{(N)} \cup \{\mu + \varepsilon_1 : \mu \in \mathcal{P}_{n-1}^{(N)}\}$; indeed each $\mu \in \mathcal{P}_{n-1}^{(N)}$ contributes

$$\frac{\mu!\left(\kappa+\frac{1}{2}\right)_\mu m_\mu(1^N)}{(\mu+\varepsilon_1)!\left(\kappa+\frac{1}{2}\right)_{\mu+\varepsilon_1} m_{\mu+\varepsilon_1}(1^N)} = \frac{1}{(\mu_1+1)\left(\kappa+\frac{1}{2}+\mu_1\right)\#\{j : \mu_j = \mu_1\}}.$$

Note that $m_\mu(1^N) = N!/\prod_{s\geqslant 0}(\#\{j : \mu_j = s\})!$ so the change from $m_\mu(1^N)$ to $m_{\mu+\varepsilon_1}(1^N)$ is the replacement of $s!$ by $1!(s-1)!$ where $s = \#\{j : \mu_j = \mu_1\}$ (except when $\mu = (0)$ in which case $m_\mu(1^N)/m_{\mu+\varepsilon_1}(1^N) = \frac{1}{N}$; this only affects the vacuous equation $\widetilde{D}_1 = 1$). The other part of the expression for $\widetilde{D}_n$ is a product with $\sum_{i=1}^n d(n-i,N) = \#\mathcal{P}_{n-1}^{(N)}$ terms (and if $\mu \in \mathcal{P}_{n-1}^{(N)}$ with $\mu_1 - \mu_2 + 1 = i$ then $\mu \in \widetilde{\mathcal{P}}_{n-i}^{(N)} + (i-1)\varepsilon_1$, $1 \leqslant i \leqslant n$). □

The determinant of the Gram matrix for $\langle\cdot,\cdot\rangle_S$ is an easy consequence by proposition 6:

$$\det(\langle h_\lambda, h_\mu\rangle_S)_{\lambda,\mu\in\widetilde{\mathcal{P}}_n^{(N)}} = \left(2^{2n}\left(\frac{N}{2}+N\kappa\right)_{2n}\right)^{-d(n,N)} \widetilde{D}_n.$$

When $N \geqslant n$ the products in $\widetilde{D}_n$ are over all partitions $\mathcal{P}_n$, $\mathcal{P}_{n-1}$ and the inner-product formula for $\langle h_\lambda, h_\mu\rangle_h$ in proposition 6 can be considered with N as indeterminate because

$$m_\sigma(1^N) = m_\sigma(1^{l(\sigma)})\binom{N}{l(\sigma)} = m_\sigma(1^{l(\sigma)})(-1)^{l(\sigma)}(-N)_{l(\sigma)}/l(\sigma)!$$

being a polynomial in N, for $\sigma \in \mathcal{P}$ and $N \geqslant l(\sigma)$. So the formula for $\widetilde{D}_n$ is an identity in the variables κ, N (for $N \geqslant n$). It would be interesting if there were an orthogonal basis for $\mathbb{H}_{2n}^B$ so that for each element f the squared norm $\langle f, f\rangle_h$ is a product of integral powers of factors linear in κ; this is almost suggested by the nice form of $\widetilde{D}_n$ but so far such a basis has not been found. The author's suspicion is that the problem is the lack of a sufficiently large set of commuting self-adjoint operators on $\mathbb{H}_{2n}^B$.

4. Concluding remarks

As mentioned in the introduction there is an orthogonal basis for the B_N-invariant wavefunctions in terms of the products of generalized Hermite polynomials in Cartesian coordinates. These are orthogonal for the measure $|t|^{2\kappa}\,\mathrm{e}^{-t^2}\,\mathrm{d}t$ on $\mathbb{R}$. For $\alpha \in \mathbb{N}_0^N$ let $\psi_\alpha(x) = \prod_{i=1}^N \left(L_{\alpha_i}^{(\kappa-1/2)}\left(\omega x_i^2\right)|x_i|^\kappa\right)\exp\left(-\omega\frac{\|x\|^2}{2}\right)$ then $\mathcal{H}\psi_\alpha = \omega(2|\alpha| + N(2\kappa+1))\psi_\alpha$ (for the Hamiltonian $\mathcal{H}$ defined in (1.1)). The functions ψ_α are pairwise orthogonal and $\int_{\mathbb{R}^N}|\psi_\alpha(x)|^2\,\mathrm{d}x = \omega^{-N(\kappa+1/2)}\Gamma\left(\kappa+\frac{1}{2}\right)^N\frac{(\kappa+\frac{1}{2})_\alpha}{\alpha!}$. The resulting B_N-invariant basis $\left\{\sum_{\alpha^+=\lambda}\psi_\alpha : \lambda \in \mathcal{P}^{(N)}\right\}$ is orthogonal but does not involve spherical harmonics.

There is an isometry between the space of $\mathbb{Z}_2^N$-invariant polynomials with the inner product $\langle\cdot,\cdot\rangle_S$ and polynomials with the L^2 inner product for the measure $\left(\Gamma\left(\frac{N}{2}+N\kappa\right)\middle/\Gamma\left(\kappa+\frac{1}{2}\right)^N\right)\prod_{i=1}^N y_i^{\kappa-1/2}\,\mathrm{d}y_1\cdots\mathrm{d}y_{N-1}$, on the simplex $\left\{y \in \mathbb{R}^N : \sum_{i=1}^N y_i = 1, y_i \geqslant 0 \text{ each } i\right\}$, induced by the correspondence $y = \left(x_1^2,\ldots,x_N^2\right)$. Then $\mathbb{H}_{2n}^0$ is isomorphic to the space of polynomials of degree n orthogonal to all polynomials of lower degree. The basis $\left\{h_\lambda : \lambda \in \widetilde{\mathcal{P}}_n^{(N)}\right\}$ maps on to a basis for the polynomials symmetric in $(y_1,\ldots,y_N)$.

If one gives up the B_N-invariance then there is no problem in constructing a nice orthogonal basis for $\mathbb{H}_n$. This basis consists of products of Jacobi polynomials; a conceptual derivation in terms of simultaneous eigenfunctions of a set of commuting self-adjoint operators can be found in [4, theorem 2.8].

In this paper, we constructed a basis for B_N-invariant spherical harmonics by introducing a new basis for the symmetric functions. The determinant of the Gram matrix of the basis was explicitly evaluated. Long ago it was found that the wavefunctions of electrons in a crystal with cubical symmetry have (energy level) degeneracies for $n \geqslant 12$; in our notation $\widetilde{\mathcal{P}}_i^{(3)} = \{(1,1)\}, \{(1,1,1)\}, \{(2,2)\}, \{(2,2,1)\}$ for $i = 2,3,4,5$, respectively but $\widetilde{\mathcal{P}}_6^{(3)} = \{(2,2,2),(3,3)\}$. We have not found a natural and constructive way of orthogonally decomposing $\mathbb{H}_{2n}^B$ when $\dim\mathbb{H}_{2n}^B > 1$; a candidate for such a decomposition is the self-adjoint operator $\sum_{1\leqslant i<j\leqslant N}(x_i\mathcal{D}_j - x_j\mathcal{D}_i)^4$ but its eigenvalues are irrational over the field $\mathbb{Q}(\kappa)$ of rational functions in κ. Yet the methods in this paper should give some insight into the problem of constructing invariant harmonics for the general B-type spin Calogero–Moser model.

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